

# A filtration question on Belyĭ pairs and dessins

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## Abstract

A Belyĭ pair is a holomorphic map from a Riemann surface to  $S^2$  with additional properties. A dessin d'enfants is a bipartite graph with additional structure. It is well known that there is a bijection between Belyĭ pairs and dessins d'enfants.

Vassiliev has defined a filtration on formal sums of isotopy classes of knots. Motivated by this, we define a filtration on formal sums of Belyĭ pairs, and another on dessin d'enfants. We ask if the two definitions give the same filtration.

## 1 Introduction

First, we recall some definitions [2, 3]. A *Belyĭ pair* is a Riemann surface  $C$  together with a holomorphic map  $f : C \rightarrow S^2 = \mathbb{C} \cup \{\infty\}$  to the Riemann sphere, such that  $f'(p)$  is non-zero provided  $f(p)$  is not 0, 1 or  $\infty$ . (Belyĭ proved that given  $C$  such an  $f$  can be found iff  $C$  can be defined as an algebraic curve over the algebraic numbers.)

A *dessin d'enfants*, or *dessin* for short, is a graph  $G$  together with a cyclic order of the edges at each vertex, and also a partition of the vertices  $V$  into two sets  $V_0$  and  $V_1$  such that every edge joins  $V_0$  to  $V_1$ . Necessarily,  $G$  must be a bipartite graph. Traditionally, the vertices in  $V_0$  and  $V_1$  are coloured black and white respectively.

It is easy to see that a Belyĭ pair gives rise to a dessin, where  $V_0 = f^{-1}(0)$ ,  $V_1 = f^{-1}(1)$ , and the edges are the components of the inverse image  $f^{-1}([0, 1])$  of the unit interval in  $\mathbb{C}$ . The cyclic order arises from local monodromy around the vertices.

A much harder result, upon which our definitions rely, is that up to isomorphism every dessin arises from exactly one Belyĭ pair, or in other words that there is a bijection between isomorphism classes of Belyĭ pairs and dessins.

## 2 Definitions

**Definition 1** (Belyĭ object). *A Belyĭ object  $B$  consists of  $((B_C, B_f), B_D)$  where  $(B_C, B_f)$  is a Belyĭ pair and  $B_D$  is the associated dessin (or vice versa for the dessin and the pair).*

**Definition 2** (Vassiliev space). *The Vassiliev space  $V = V_{\mathbb{C}}$  (for Belyĭ objects) is the vector space over  $\mathbb{C}$  which has as basis the isomorphism classes of Belyĭ objects.*

Clearly, when an edge is removed from a dessin then it is still a dessin. Suppose  $D$  is a dessin, and  $T$  is a subset of its edges. We will use  $D \setminus T$  to denote the dessin so obtained. This same operation can also be applied to a Belyĭ object  $B$ , even though computing the associated curve  $(B \setminus T)_C$  from  $B_D$  and  $T$  might be hard.

We will now define one or two filtrations of  $V$ .

**Definition 3** (Dessin with  $d$  optional edges). *Let  $D$  be dessin and  $S$  a  $d$ -element subset of  $D$ . Each subset  $T$  of  $S$  determines a dessin  $S \setminus T$  and hence a Belyĭ object  $B_{S \setminus T}$ . Let  $|T|$  denote the number of edges in  $T$ . Use*

$$B_S = \sum_{T \subseteq S} (-1)^{|T|} B_{S \setminus T}$$

*to define a vector  $B_S$  in  $V$ , which we call the expansion of a dessin with  $d$  optional edges.*

**Definition 4** (Dessin filtration). *Let  $V_{D,d}$  be the span of the expansions of all dessins with  $d$  optional edges. The sequence*

$$V = V_{D,0} \supseteq V_{D,1} \supseteq V_{D,2} \supseteq V_{D,3} \dots$$

*is the dessin filtration of  $V$ .*

We can also think of a Belyĭ object as a map  $f : C \rightarrow S^2$  (with special properties). Let  $(C_1, f_1)$  and  $(C_2, f_2)$  be Belyĭ pairs. Then there is of course a map

$$g : C_1 \times C_2 \rightarrow S^2 \times S^2.$$

Let  $\Delta \subset S^2 \times S^2$  denote the diagonal, and let  $C$  denote  $g^{-1}(\Delta)$ , and  $f$  the restriction of  $g$  to  $C$ . In general

$$f : C \rightarrow \Delta \cong S^2$$

will not be a Belyĭ pair. There are two possible problems. The first is that  $C \subset C_1 \times C_2$  might have self intersections or be otherwise singular. If this happens, we replace  $C$  by its resolution, which is unique.

The second problem is more interesting. It might be that  $f$  has critical points not lying above the special points 0, 1 and  $\infty$ . This problem cannot be avoided. However, the above discussion does show that there is product, which we will denote by ‘ $\circ$ ’, on holomorphic branched covers of  $S^2$ .

**Definition 5** (Product filtration). *Let  $W$  be the vector space with basis isomorphism classes of branched covers of  $S^2$ . We set  $W_n$  to be the span of all products of the form*

$$(A_1 - B_1) \circ (A_2 - B_2) \circ \dots \circ (A_n - B_n)$$

*for  $A_i$  and  $B_i$  basis vectors of  $W$ . Clearly, the  $W_n$  provide a filtration of  $W$ .*

**Definition 6** (Belyĭ filtration). *The induced filtration of  $V$  defined by  $V_{B,n} = W_n \cap V$  is called the Belyĭ filtration of  $V$ .*

### 3 Questions

**Question 1.** *Are the two filtrations  $V_D$  and  $V_B$  equal?*

If so, then we have also answered the next two questions.

**Question 2.** *The absolute Galois group acts on Belyĭ pairs, and preserves the Belyĭ filtration. Does this action also preserve the dessin filtration?*

**Question 3.** *Because the dessins with  $d$  edges, all of which are optional, span  $V_d/V_{d+1}$ , the dessin filtration has finite dimensional quotients. Does the Belyĭ filtration have finite dimensional quotients?*

Investigating the last two questions might help us answer the first. They might also be of interest in their own right.

### References

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- [2] G. V. Belyĭ, Another proof of the three points theorem, *Subornik: Mathematics* 193 (2002), 329–32.
- [3] Leila Schneps, ed, *The Grothendieck Theory of Dessins d’Enfants*, London Math. Soc. Lecture Note Ser., vol 200, Cambridge Univ. Press 1994.